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Momentum-Balance Aspects of Free-Settling Theory. I. Batch Thickening

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Abstract

The theory of batch gravity thickening of an initially free-settling slurry is considered, taking into account the momentum-balance relationships for the system. It is concluded that graded-concentration zones which develop during the process must lie in the compression concentration range; retarding forces necessary to produce increase in concentration are not present in free settling.

INTRODUCTION

Since the publication of the classical paper of Coe and Clevenger (*1*), it has usually been considered that the concentration of a slurry can fall in one of two basic ranges: "free settling" and "compression." In the free-settling range the flocs (or "particles," if the material is not flocculated) do not exert forces on each other, while in compression they do. The boundary between these two ranges is the "critical concentration," which is conceived as the concentration at which flocs just touch each other. Below the critical concentration the slurry is in free settling; above, it is in compression.

During batch settling of an initially uniform, dilute (i.e., free-settling)

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slurry, Coe and Clevenger observed that four zones are usually formed. Labeled A, B, C, and D from the top, these were identified as: clear-liquid zone, initial-concentration zone, graded-concentration zone, and sediment zone. They assumed that Zone C (and, of course, Zone B) is in free-settling, while Zone D is in compression. (For Zone A to be completely clear, it is necessary that the initial concentration is not so low that segregation of different-sized particles occurs.) After settling is complete, only Zones A and D remain. Later writers adhere to this qualitative view of the process of batch thickening.

Coe and Clevenger assumed that in free settling the settling velocity of the flocs (in a given slurry) is a function only of the solids concentration. This assumption was the basis of their method (still used) for determining the area required for a given continuous thickening operation. Much later, Kynch (*2*) used the same assumption as the basis of a detailed analysis of batch thickening. This analysis predicted the formation of a graded-concentration zone (Zone C) between the initial-concentration and sediment zones, under certain circumstances. (Coe and Clevenger had remarked that Zone C does not always form.)

More recently, detailed analyses of slurry behavior in compression have appeared (e.g., Refs. *3–5*). In addition to material balances, these analyses have involved the momentum balance equation; that is, consideration of the forces acting in the system. This is a consideration which is not involved in the analysis of free settling when the settling velocity is assumed to be determined by the concentration. The purpose of the present work is to reconsider the theory of batch thickening for cases where the initial slurry concentration is in the free-settling range, taking into account the momentum balance relationships for the system. This approach reveals aspects of the theory which have not been considered previously, and raises doubts about the logical consistency of presently accepted analyses.

BASIC ASSUMPTIONS AND EQUATIONS

Throughout the discussion, the following assumptions will be made:

(1) The container has constant cross-sectional area.

(2) The slurry properties are uniform and constant; that is, the flocs are of uniform size (no segregation of different-sized particles) and solid and liquid properties are constant (isothermal conditions).

(3) The flow is vertical, and horizontally uniform (negligible wall effect).
(4) The forces which can act on the solid particles are gravity (allowing for buoyancy), liquid drag due to motion relative to the liquid, and interaction forces exerted by adjacent solids.
(5) The slurry can be treated as a continuum; that is, a continuous liquid phase and a continuous solid phase which interact with each other.

Assumption 1 can easily be removed, if desired, but this is not necessary in the present discussion. The other assumptions are quite standard, although they are not always stated explicitly. The phrase "can act" should be noted in Assumption 4, indicating that all three forces are not necessarily acting at every point (the first, of course, always acts).

Assumption 5 leads to the definitions given under *Symbols*. In particular, the following are noted:

(a) Fluxes (ϕ) are defined throughout as volumetric fluxes; that is, volumetric flow rate divided by total cross-sectional area, which is the same as "superficial velocity." There are three fluxes: solid, liquid, and total. The total flux is also the volume-average velocity of the slurry.
(b) Solids concentration is defined as volume fraction (f), and thus liquid volume fraction is everywhere equal to $(1 - f)$.
(c) The velocity (relative to the container) of each phase is calculated as its flux divided by the local volume fraction occupied by the phase; e.g., solids velocity, $v = \phi/f$. The velocity of the solids (u) relative to the slurry is equal to $v - \phi_t$.
(d) The interaction force between adjacent parts of the solid phase is described by compressive stress, τ, based on total cross-sectional area; that is, the force acting between adjacent solid layers per unit of total cross section.

The basic equations describing the thickening process are the total material balance, the material balance for the solids (or the liquid), and the momentum balance for the solids. The material balances are simply volume conservation statements (volume is conserved since both liquid and solids have constant density). The solids momentum balance is a statement of Newton's second law of motion. The momentum balance for the liquid is not required unless the forces acting on the bottom of the container (or the power requirements for the sludge pump in a continuous thickener) are to be determined. No energy balance is required since isothermal conditions are assumed.

The equations are listed in Table 1 and derived in Appendix A. (Most of these equations, or their equivalents, have been given by previous writers.) Only the total material balance equation is algebraic; the other equations are differential. The total material balance states simply that the total volumetric flux (ϕ_t) is the same at all levels, and this leads to Eq. (1) which is the relation between the velocity of the solids (u^*) relative to the liquid, and the velocity of the solids (u) relative to the slurry. The two differential equations can be written in a variety of forms, depending on whether they apply to a plane which is stationary (Eulerian), or moving with the local solids velocity (Lagrangian), or having fixed concentration (Kynchian), etc. The equations labeled "general" are derived without specifying the type of plane being considered, and have no physical meaning as they stand. However, the equations for Eulerian, Lagrangian,

TABLE 1

Basic Continuum Equations

Total material balance	$u = (1 - f)u^*$	(1)
Solids material balance		
General	$\frac{df}{dt} = -\left(\frac{\partial\phi}{\partial x}\right)_t + \frac{dx}{dt}\left(\frac{\partial f}{\partial x}\right)_t$	(2)
Eulerian	$\left(\frac{\partial f}{\partial t}\right)_x = -\left(\frac{\partial\phi}{\partial x}\right)_t$	(3)
Lagrangian	$\left(\frac{\partial f}{\partial t}\right)_s = -f\left(\frac{\partial v}{\partial x}\right)_t$	(4)
Kynchian	$\left(\frac{\partial x}{\partial t}\right)_f = \left(\frac{\partial\phi}{\partial f}\right)_t$	(5)
Discontinuity	$\frac{dx}{dt} = \frac{\phi_2 - \phi_1}{f_2 - f_1}$	(6)
Solids momentum balance		
General	$\rho\frac{dv}{dt} = F_g + F_d - \frac{1}{f}\left(\frac{\partial\tau}{\partial x}\right)_t + \rho\left(\frac{dx}{dt} - v\right)\left(\frac{\partial v}{\partial x}\right)_t$	(7)
Eulerian	$\rho\left(\frac{\partial v}{\partial t}\right)_x = F_g + F_d - \frac{1}{f}\left(\frac{\partial\tau}{\partial x}\right)_t - \rho v\left(\frac{\partial v}{\partial x}\right)_t$	(8)
Lagrangian	$\rho\left(\frac{\partial v}{\partial t}\right)_s = F_g + F_d - \frac{1}{f}\left(\frac{\partial\tau}{\partial x}\right)_t$	(9)
Kynchian	$\rho\left(\frac{\partial v}{\partial t}\right)_f = F_g + F_d - \frac{1}{f}\left(\frac{\partial\tau}{\partial x}\right)_t + \rho\left\{\left(\frac{\partial\phi}{\partial f}\right)_t - v\right\}\left(\frac{\partial v}{\partial x}\right)_t$	(10)
Discontinuity	$\tau_2 - \tau_1 = \rho\frac{f_1 f_2}{f_2 - f_1}(v_2 - v_1)^2$	(11)

and Kynchian planes derive directly from the general equations by applying the appropriate constraint. For example, the Eulerian equations are obtained by substituting $dx/dt = 0$ in the general equations.

It is also convenient in many cases to use as spatial coordinate the volume of solids per unit area (s) below a Lagrangian datum plane, instead of the distance (x) below an Eulerian datum plane, and to use the dilution ($r = 1/f$) instead of the volume fraction (f). The equations in Table 1 have been written in terms of x and f. The equivalents in terms of s and r can be obtained by use of $df = -dr/r^2$, $ds = f\,dx$ at constant t, and $ds/dt = f(dx/dt - v)$. The equations applying to a discontinuity, if such occurs in the system, are also given in Table 1, and are discussed in the next section of the paper.

It will be noticed that the liquid pressure gradient does not appear in the solids momentum balance, although the pressure does act, of course, on the solids. The pressure gradient can be expressed as the sum of three components:

(1) Hydrostatic pressure gradient $\rho_l g$.
(2) Gradient required because of the drag on the solid phase, $-F_d f$ [the factor f is required because F_d is defined as the drag force per unit volume of solids].
(3) Gradient required to accelerate the liquid.

The effects of the first two of these are included in the solids momentum balance; the first as the buoyancy correction in F_g, and the second as the drag term F_d. The third represents the unbalanced force required to accelerate the liquid under unsteady-state conditions. This does not result in any force on the solids. If it did, the solids would exert an equal and opposite force on the liquid (third law of motion), thereby counteracting the force which, by definition, is unbalanced.

The definitions of "free-settling," "compression," and "critical concentration" given in the Introduction in no way affect the validity of the equations. In free settling τ is simply set equal to zero. The equations in Table 1 cannot be solved alone; also required are equations describing slurry properties. Equations relating the drag force and compressive stress to the condition of the slurry are needed, and these will be introduced below, when required. To this point the assumptions made are the five listed above.

In connection with the basic assumptions, it is important to note that the horizontal-uniformity assumption (No. 3) means that effects such as short-circuiting and channeling are not considered, while the constant properties assumption (No. 2) exludes effects due to changes in degree of flocculation, etc. during the settling process. The discussion deals solely with what might be termed "basic" theory, and comparisons will only be made with previous analyses which were based on the same assumptions as listed above.

CONDITIONS AT A DISCONTINUITY

In the following discussion the possibility of the formation of discontinuities in the slurry arises, and so the conditions existing in such a situation need to be considered. At a discontinuity there is a jump in either concentration or velocity, and usually both. As shown by the material balance equation (Eq. 6), a plane of discontinuity cannot be stationary, except where there is a discontinuity in concentration but not in solids flux, which is an unlikely situation.

Kynch (*2*) used Eq. (6) in his analysis, but not the momentum balance equation (Eq. 11) since force action was not considered. In deriving the latter equation it is assumed that the drag term (F_d) is related to the concentration and solid and liquid velocities algebraically (rather than differentially). The gravitational term also satisfies this condition (it is a constant), and the result is that no gravitational or drag term appears in the momentum balance across a discontinuity. This is due to the fact that the total gravitational and drag force acting in a layer is proportional to the quantity of solids, and in a discontinuity the quantity of solids is zero.

However, there is an additional component of the drag which does not vanish in the momentum balance for a discontinuity because it depends on the rate of change of the solid–liquid relative velocity. This is the "virtual-mass" term, which is effective under unsteady-state conditions but is zero when the relative velocity is constant. This effect does not appear to have been considered by writers on slurry thickening, but it has been the subject of some discussion in the literature on fluidization (e.g., Refs. *6–8*). The exact form of the virtual-mass term is unknown at present, but there is agreement on the general form, and it effectively increases the magnitude of the inertial (acceleration) term in the momentum balance. To avoid unnecessarily complicating the following discussion, we will not include any specific virtual-mass term in the momentum equations, but simply note at this point that the effect is equivalent to an increase in the inertial term. This is all that is needed in the following discussion. More detailed discussion of the virtual-mass effect is given in Appendix B. Henceforth the drag term F_d will be assumed to include only the "ordinary," algebraic drag, while reference to the inertial term will be assumed to include the virtual-mass term.

The absence of gravitational and drag terms from Eq. (11) is important to the following discussion since it shows the momentum balance for a discontinuity to be a balance between inertial and compressive effects, no gravitational or drag terms being involved. Since there are no compressive effects in free settling, by definition, the balance further shows that at least one side of a discontinuity must be in compression (except in the unlikely event of there being no change in velocity). To produce a change in velocity, forces are required, and to produce an instantaneous change in velocity the only forces available are interparticle forces. The usual situation is that the solids increase in concentration and decrease in velocity as they pass downward through the discontinuity. Thus the solids undergo impact retardation as they pass through the discontinuity, and the solids must be in compression below the discontinuity in order to be able to transmit the necessary impulsive force.

Objection might be raised to the above on the grounds that a true mathematical discontinuity cannot occur in a physical system; what can occur is a rapid change over a small but finite distance, and so gravitational and drag forces could account for the velocity change. However, a true discontinuity is consistent with the assumption that the slurry can be treated as a continuum. Both are approximations, and the former is a consequence of the latter. On the individual-particle level, change in velocity certainly can only occur through a finite distance, but if the continuum approximation is satisfactory (which it is usually assumed to be for fine particles) and it leads to the existence of a discontinuity, then this indicates that the contribution to the velocity change of gravity and drag is small and that Eq. (11) is correct.

Thus consideration of the momentum balance across a discontinuity leads to the conclusion that the only terms in the equation are the inertial and compressive terms, and that the concentration on no more than one side can be in the free-settling range.

CONVENTIONAL SETTLING VELOCITY ASSUMPTION

As noted above, the usual assumption applied to the free-settling concentration range is that the solids settling velocity is determined solely by the local concentration. The settling velocity can be taken relative to the liquid or relative to the slurry; if one is a function of concentration, so is the other, from Eq. (1). For the purposes of the present discussion, it is necessary to consider the implication of this assumption in terms of the forces acting on the solids.

Neither Coe and Clevenger nor Kynch discussed this question in any detail, and it has seldom been mentioned by other writers. Nevertheless, the origin of the assumption that settling velocity is a function of concentration is clear. Considering first a single small particle starting from rest in a stationary fluid, it is known that it will very rapidly accelerate to its terminal velocity. Initially, the only force acting on the particle is the gravitational force (corrected for buoyancy), and this determines the initial rate of acceleration of the particle. As the particle velocity increases, the fluid drag increases and the net force on the particle decreases. When the terminal velocity is reached, the gravitational force is balanced by the drag force and there is no further acceleration. Since the acceleration of the particle is very rapid (in the case of a small particle) and occurs over a very small distance, the acceleration effect can usually be neglected. That is, to calculate the time required to settle a distance of the magnitude which is usually involved in a thickening operation, the particle velocity can be taken from the start as equal to the terminal velocity, with negligible error.

If a slurry is now considered instead of a single particle, the drag force on any particle will be greater, for the same velocity relative to the liquid, due to the proximity of the other particles. The higher the solids concentration the greater this effect should be, and the drag is expected to be a function of the local concentration and the velocity relative to the liquid. If the acceleration effects are again assumed to be negligible, the gravitational and drag forces are balanced at each point in the slurry. Since the former force is a constant and the latter is a function of relative velocity and concentration, this implies a relation between the relative (or "settling") velocity and the concentration. That is, at each point in the slurry the settling velocity is determined only by the concentration.

Relating this result to the equations in Table 1, the Lagrangian momentum balance (Eq. 9) is the relevant equation. Since free-settling is being considered, τ is zero by definition. Since the inertial effect is neglected, $(\partial v/\partial t)_s$ is taken as zero. Equation (9) reduces to $F_g + F_d = 0$. Since F_g is constant, and F_d is a function of u and f, this gives a relation between u and f.

Thus the assumption that the settling velocity is a function of concentration in free settling is equivalent to assuming:

(6) The drag force on the solid at any point is a function of its concentration and its velocity relative to the liquid.
(7) Inertial effects may be neglected everywhere.

As mentioned above, few previous writers have discussed the $u = u(f)$ assumption in terms of forces. However, the few who have discussed the matter place the same interpretation on the assumption as given here (e.g., Ref. *9*). Kynch (*2*) remarked that the assumption would only be expected to be valid "when the speed of propagation (of concentration changes) is relatively slow or the damping is great," implying small acceleration effects.

INCOMPRESSIBLE SLURRIES

When a batch of slurry is allowed to stand in a cylinder, it is clear that all the solids will be present ultimately in a stationary sediment resting on the bottom. Since the sediment is supported by the cylinder bottom and there is no drag force on the solids because there is no motion, it is also clear that each layer of solids is supported directly by the solids below; that is, the sediment is in compression. During the settling process the sediment will build up on the bottom of the cylinder and the solids in each layer in it will be at least partially supported by compressive forces exerted by the solids below, while also being partially supported by drag forces. Thus it is not possible to have a batch thickening process in which the slurry is in free settling everywhere and at all times.

The nearest approach to a purely free-settling batch sedimentation process is through the concept of an "incompressible" slurry. This idealized slurry forms a sediment which is so strong that lower layers are not detectably compressed to higher concentrations by the weight of the layers above. The sediment has a uniform concentration which is the critical concentration. When particles first come into contact they are at the critical concentration, and in an incompressible slurry the particles never achieve a higher concentration because the application of a compressive stress, due to build-up of further sediment above, causes no further increase in concentration. Thus in an incompressible slurry the critical concentration is also a maximum concentration, and the uniform concentration of the sediment. Also, there is no movement in the sediment, so that the settling velocity is zero at the maximum concentration. At concentrations less than the maximum, the slurry is in free settling and so, under Assumptions 6 and 7, the settling velocity is a function of the concentration. Thus, for an incompressible slurry, the usual assumption is that the settling velocity is a function of concentration, being zero at the maximum concentration.

This statement was the basis of Kynch's analysis (*2*) which is, therefore, effectively based on Assumptions 1 to 7, and the assumption of sediment incompressibility.

KYNCH'S THEORY

Kynch first showed that each concentration moves with a fixed, characteristic velocity. This result follows directly from Eq. (5). The solids flux (ϕ_r) relative to the flux induced by the bulk motion is given by $\phi = \phi_r + f\phi_t$. Since ϕ_t does not vary with depth, substitution for ϕ in Eq. (5) gives $(\partial x/\partial t)_f - \phi_t = (\partial \phi_r/\partial f)_t$. The left-hand side of this equation is the velocity of a plane of fixed concentration relative to the slurry volume-average velocity. Under the assumption that u is a function of f, ϕ_r (which is equal to fu) is also a function of f, and so the right-hand side of the equation reduces to $d\phi_r/df$, also a function of f. Thus a plane of fixed concentration moves relative to the slurry with a fixed velocity which is characteristic of the concentration. (In batch thickening ϕ_t is zero, so that the velocity relative to the slurry is the same as the velocity relative to the container.)

Kynch then considered the batch thickening of an incompressible slurry with initially uniform concentration. In accord with the basic assumption $u = u(f)$, in the initial condition the solids will be settling with the velocity corresponding to the initial concentration, except on the bottom where the velocity is zero. Since the velocity is zero on the bottom, the concentration there must be the maximum concentration. Thus in the initial condition there is a differentially deep sediment at maximum concentration on the bottom, and there is a discontinuity in both concentration and velocity between the sediment and the initial concentration zone.

Kynch took the view (not unreasonable) that the initial concentration discontinuity at the bottom of the cylinder could be considered to contain all concentrations between the two limits of the discontinuity (i.e., between the initial and maximum concentrations). Since his basic assumption led to the result that each of these concentrations moved relative to the cylinder with a characteristic velocity, he could show the conditions under which the initial discontinuity would change into a continually expanding zone of graded concentration lying between the uniform sediment and initial-concentration zones. This zone was interpreted, of course, as Coe and Clevenger's Zone C. Although it was soon recognized that the assumption $u = u(f)$ does not generally apply to real slurries, which form compressible sediments, Kynch's analysis appeared to go a long way toward explaining observed batch thickening behavior.

However, when this analysis is reconsidered, taking into account the momentum balance relationship for the system, one is led to doubt that the assumption $u = u(f)$ is generally valid even for an incompressible slurry. Kynch's argument centers on the behavior of the initial discontinuity between initial-concentration and sediment zones. As discussed above, under the heading Conditions at a Discontinuity, the momentum balance across a discontinuity (Eq. 11) contains no gravity or drag terms, while Kynch's basic assumption is equivalent to assuming a balance between these two forces. Clearly, an analysis cannot be based on the assumption of a balance between two forces in a region where these forces are insignificant compared to other forces involved.

Even if the demand of Eq. (11) that compressive effects are necessary for occurrence of a discontinuity is ignored, and it is assumed that a discontinuity can occur entirely in the free-settling concentration range, the assumption $u = u(f)$ still requires neglect of inertial effects in a discontinuous change. Since inertial effects depend on rate of change of velocity, and this is infinite for material passing through a discontinuity, inertial effects cannot be assumed to be negligible in a discontinuity, no matter how small the particles. A discontinuity is not a place "where the speed of propagation (of concentration changes) is relatively small." Even if the "discontinuity" is considered to involve a rapid (but not infinitely fast) change in velocity, it is still difficult to accept that u is determined solely by a balance between gravity and drag, with negligible inertial effects.

A further difficulty with Kynch's theory arises from the fact that it predicts under certain conditions that the initial discontinuity will form a graded-concentration zone with a discontinuity above and below (*9*). The upper of these discontinuities has free-settling concentrations on each side, which Eq. (11) does not allow. Thus even if it is argued that the initial discontinuity is unstable, so that after spreading a small amount its behavior can be calculated by assuming a balance between gravity and drag, making $u = u(f)$ correct, the analysis can still lead to an inconsistency in the form of a stable discontinuity with free-settling concentrations on each side.

BATCHING THICKENING OF AN INITIALLY-UNIFORM SLURRY

The writer believes that the correct analysis of the batch thickening of an initially uniform, free-settling slurry is as follows:

Consider first an incompressible slurry, initially uniform and at rest. The solids are everywhere subject to the same initial accelerating force (gravity minus buoyancy), except for the bottom layer, which remains stationary. Except near the bottom, all solids undergo the same acceleration and attain the same velocity. As the velocity increases the drag increases, and the terminal velocity corresponding to the initial concentration is rapidly attained. At this stage the solids are subject to no net force.

At the bottom of the container the uniform (incompressible) sediment builds up as solids arrive from above. As the solids strike the top of the sediment their concentration jumps to the maximum concentration and their velocity jumps to zero. The force necessary for this jump in velocity is provided by the reaction from the bottom of the container transmitted through the sediment. The compressive stress acting on the top of the sediment is given by Eq. (11) as $\rho f_m f_i v^2/(f_m - f_i)$, where f_m = sediment concentration, f_i = initial concentration, and v = solids velocity in the free-settling zone (which rapidly approaches the terminal velocity).

The question is: "Can a concentration gradient propagate upward from the sediment into the settling zone?," and the answer is "No." For an increase in concentration to occur in any layer in the free-settling zone, a negative velocity gradient must develop (Eq. 4). This means, in turn, that each layer must be reduced in velocity as it approaches the sediment, and this requires a retarding force. Hence the question now is whether each layer can encounter a retarding force before it is retarded by impact with the sediment. Clearly, it cannot. Starting from rest, the settling layers experience an accelerating force which rapidly approaches zero as the terminal velocity is approached. Thereafter, the net force remains at zero until the layer is retarded on striking the sediment. If the solids were to experience a retarding force before reaching the sediment, it would have to be due to an increase in the drag force, since the gravitational force is constant and there is no compressive effect in free settling. For the drag to increase would require an increase in concentration (but that is the effect which is sought, and cannot simultaneously be a cause), or an increase in velocity (but that would result in a decrease in concentration). Thus there is no retardation until the sediment is reached and no increase in concentration within the settling zone. The conclusion reached is that the settling is always "Type I"; that is, during the process only two solids zones appear, the initial-concentration Zone B and the sediment Zone D. Zone D increases in depth as it is fed with solids from Zone B, and the process finishes when Zone B is exhausted.

Once again, objection might be raised to this argument by those who are not prepared to adhere strictly to the consequences of the continuum assumption. It might be argued that if one considers an individual particle approaching the sediment, its "concentration" will increase as it approaches the particles at the top of the sediment, with the result that the drag on it increases, and so it is subject to retardation before reaching the sediment. While this is a valid *noncontinuum* view, it does not necessarily show that a concentration gradient will spread upward continually into the settling zone, which is predicted by the *continuum* argument based on the $u = u(f)$ assumption. As already discussed, the discontinuity between Zones B and D is an approximation which is a consequence of the continuum approximation. An individual particle will be retarded over a finite distance, but this distance could be very small and might not increase as the process proceeds. Since the continuum analysis leads to the conclusion that the initial discontinuity between initial-concentration and sediment zones remains during the process, the logical conclusion is that the retardation distance is, in fact, small and not continually increasing.

Further support for this argument is given by a study of a simplified, noncontinuum slurry model, previously reported (*10*). In this model, individual particles were considered during sedimentation of an initially-uniform, incompressible slurry. When inertia was neglected in the equations describing the system, a graded-concentration zone spread upward from the sediment in certain cases, in agreement with Kynch's analysis. However, when inertia was included in the equations, no graded-concentration zone spread upward, only a thin, constant-thickness particle-retardation layer was formed, in agreement with the continuum arguments given above.

It should be noted that the above arguments do not deny that inertial effects are usually very small, so that retarding forces necessary for concentration increase are also small. However, small or not, a retarding force is nevertheless required to produce an increase in concentration, and is not available in free settling.

The argument is not essentially changed when extended to include slurries which form compressible sediments. In such a case, lower layers will be compressed to higher concentrations as the sediment builds up because of the increasing weight of solids above. The compressive stress acting at a given level in the sediment is determined by the total submerged weight of solids in the sediment above minus the total drag force acting on the solids above, plus small contributions from the inertial effect and the impact stress acting on the top surface of the sediment as free-settling particles enter the sediment. Thus a concentration gradient develops in the sediment, with the concentration increasing downward.

The development of a concentration gradient in the sediment has no qualitative effect on the behavior of the free-settling zone. By the same argument as given in the preceding paragraphs, there will be no development of a concentration gradient above the sediment because there is no force available for solids retardation before reaching the sediment. As the solids are retarded by impact on the top of the sediment, the stress acting on the sediment will be still given by Eq. (11); the only difference is that the solids velocity is no longer zero at the top of the sediment.

For batch thickening of a compressible slurry, it is concluded, therefore, that there will basically be only three zones: clear liquid, initial-concentration, and sediment. Only the sediment will contain a concentration gradient, as a result of the compression process, and Zones C and D of Coe and Clevenger must both be part of the compression zone. The incompressible-sediment case is simply a special case in which no gradient appears in the sediment because negligible compression occurs.

DISCUSSION AND CONCLUSIONS

The foregoing analysis of free settling, which included consideration of force action, has led to quite different conclusions from previously published analyses. The essential feature of the present argument is that it takes account of the fact that velocities cannot change without the action of forces, which, of course, is the first law of motion.

Previous analyses of free settling were based on the assumption that settling velocity is a function of concentration. For present purposes it was necessary to interpret this assumption in force terms, and it was concluded above, under the heading Conventional Settling Velocity Assumption, that it is equivalent to Assumptions 6 and 7, stated there. The writer does not know of any other interpretation which can be placed on the $u = u(f)$ assumption, but if there is another reasonable interpretation, this could alter the conclusions reached. The other assumptions made in the present analysis were also made in previous analyses.

It is realized that many readers will be unable to accept the analysis presented purely on its logical merits. However, the writer has no choice but to ask readers to do this, since there appears to be no present experimental data which can show whether the present or previous analyses are correct, and it seems that it will not be an easy matter to obtain such data. The basic difference from previous results is that it is concluded that concentration gradients in batch thickening can only develop in the compression zone, not in the free-settling zone. Hence the experimental task is to determine whether compressive stresses exist in the graded-concentration zone or not, and the writer does not know of any direct and accurate method for doing this. Even in a slurry of glass spheres the short graded-concentration zones observed (*9*) could be due to compressive effects. In cubical packing the concentration obtained with identical spheres is 0.524, while in tetragonal packing it is 0.741, and in random packing it is between these values. It is clearly possible for spheres to come into sliding contact before they reach their final concentration, so that a compressive effect could exist even in this apparently incompressible-sediment case.

A further conclusion from the present study is that Coe and Clevenger's Zones C and D must both be part of the compression zone. If this is correct, why then does the compression zone appear to the eye in some cases as two distinct zones? A reasonable answer to this can be found from consideration of experimentally measured concentration versus depth curves for sediments at the completion of batch settling (e.g., Ref. *11*, Fig. 2). Such curves show that, starting from the top of the sediment, the concentration initially increases very rapidly with depth but that the rate of increase with depth rapidly drops to a very low value. Since compressive stress increases with depth, this indicates that when the flocs first come into contact (the critical concentration), the structure is very easy to compress, but as the concentration increases the ease of compression rapidly decreases until the sediment is almost incompressible. Thus, in the upper part of the sediment, during the settling process, the concentration will change rapidly and could appear to the eye as a graded-concentration zone, while the lower part appears to be a separate uniform-concentration zone.

Thus, based on Assumptions 1 to 6, the main conclusion reached by the theoretical argument given here is that, starting with a uniform suspension, there is no way in which a concentration gradient can be formed in the free-settling zone. When a concentration gradient is formed it necessarily must be in the compression zone, because it is only in this zone that retarding forces, necessary to produce a concentration increase, are available.

APPENDIX A

Derivation of Equations

Total Material Balance

Since both solid and liquid phases have constant density, there is no change in volume within the system, so that the total volumetric flux is the same at every level. That is,

$$\phi + \phi_l = \phi_t = \text{constant}$$

i.e.,

$$fv + (1 - f)w = \phi_t$$

i.e.

$$w = \frac{\phi_t - fv}{1 - f}$$

Therefore, the velocity of the solids relative to the liquid is given by

$$u^* = v - w = v - \frac{\phi_t - fv}{1 - f}$$

$$= \frac{v - \phi_t}{1 - f} = \frac{u}{1 - f}$$

which leads directly to Eq. (1).

Solids Material Balance

Consider two differentially-spaced, horizontal planes, labeled 1 and 2, and use subscripts 1 and 2 to denote values at these planes. At this stage, no specification is made to identify the planes as Eulerian, Lagrangian, Kynchian, etc.

The volumetric solids balance (input rate = output rate + accumulation rate) on the differential layer defined by planes 1 and 2 is

$$f_1\left(v_1 - \frac{dx_1}{dt}\right) = f_2\left(v_2 - \frac{dx_2}{dt}\right) + \frac{d}{dt}\left[\int_1^2 f\,dx\right]$$

The integral is approximated by the trapezoidal rule as $0.5(f_2 + f_1)(x_2 - x_1)$. Expanding the derivative of this and rearranging the equation gives

$$(\phi_2 - \phi_1) - \frac{f_2 - f_1}{2}\frac{d(x_2 + x_1)}{dt} + \frac{x_2 - x_1}{2}\frac{d}{dt}(f_2 + f_1) = 0 \qquad (12)$$

Dividing throughout by $(x_2 - x_1)$ and taking the limit as $(x_2 - x_1) \to 0$ gives

$$\frac{df}{dt} = -\left(\frac{\partial \phi}{\partial x}\right)_t + \frac{dx}{dt}\left(\frac{\partial f}{\partial x}\right)_t \qquad (2)$$

which is the "general" material balance equation. As explained in the text, the Eulerian, Lagrangian, and Kynchian forms can be obtained directly from this by making the appropriate substitutions.

Solids Momentum Balance

In a similar way the solids momentum balance is written for a differential layer and is

$$\rho v_1 f_1\left(v_1 - \frac{dx_1}{dt}\right) + \int_1^2 (F_g + F_d) f\,dx + \tau_1 = \rho v_2 f_2\left(v_2 - \frac{dx_2}{dt}\right) + \tau_2 + \frac{d}{dt}\left[\int_1^2 \rho v f\,dx\right]$$

Approximating both integrals by the trapezoidal rule, expanding the time derivative, and rearranging:

$$0 = -\rho(\phi_2 v_2 - \phi_1 v_1) + \left\{\frac{1}{2}F_g(f_1 + f_2) + \frac{1}{2}(f_2 F_{d2} + f_1 F_{d1})\right\}(x_2 - x_1) - (\tau_2 - \tau_1) + \frac{1}{2}\rho(\phi_2 - \phi_1)\frac{d}{dt}(x_2 + x_1) - \frac{x_2 - x_1}{2}\rho\frac{d}{dt}(\phi_2 + \phi_1) \qquad (13)$$

Dividing throughout by $(x_2 - x_1)$ and taking the limit as $(x_2 - x_1) \to 0$,

$$\rho\frac{d\phi}{dt} = f(F_g + F_d) - \frac{\partial \tau}{\partial x} + \rho\frac{dx}{dt}\left(\frac{\partial \phi}{\partial x}\right)_t - \rho\frac{d}{\partial x}(\phi v)$$

Substituting $\phi = fv$, expanding the three ϕ derivatives, and substituting for df/dt from Eq. (2):

$$\rho \frac{dv}{dt} = F_g + F_d - \frac{1}{f}\frac{\partial \tau}{\partial x} + \rho\left(\frac{dx}{dt} - v\right)\left(\frac{\partial v}{\partial x}\right)_t \tag{7}$$

which is the "general" solids momentum balance equation.

Discontinuity Equations

Writing the solids material balance for a plane of discontinuity, using subscripts 1 and 2 to distinguish the two sides of the discontinuity, and noting that there is no accumulation in a plane:

$$f_1\left(v_1 - \frac{dx}{dt}\right) = f_2\left(v_2 - \frac{dx}{dt}\right)$$

Rearranging,

$$\frac{dx}{dt} = \frac{\phi_2 - \phi_1}{f_2 - f_1} \tag{6}$$

which gives the velocity of the plane. An alternative method for deriving this equation is to consider the discontinuity as defined by two planes 1 and 2 initially a finite distance apart. The material balance is then the same as for a differential layer and leads to the Eq. (12) above. If the limit is taken as $x_1 \to x_2$ [but *not* dividing throughout by $(x_1 - x_2)$ first], the layer becomes a discontinuity and the equation becomes

$$(\phi_2 - \phi_1) - (f_2 - f_1)\frac{dx}{dt} = 0$$

leading directly to Eq. (6).

The solids momentum balance for a discontinuity is obtained in a similar way to the above by letting $x_1 \to x_2$ in Eq. (13) above. Assuming that F_d contains no spatial derivatives and so remains finite as $x_1 \to x_2$, the equation becomes

$$\tau_2 - \tau_1 = -\rho(\phi_2 v_2 - \phi_1 v_1) + \rho(\phi_2 - \phi_1)\frac{dx}{dt}$$

Substituting for dx/dt from Eq. (6) and rearranging, Eq. (11) is obtained.

APPENDIX B

Virtual-Mass Drag Term

When a particle is accelerating through a fluid it experiences a greater drag at a given velocity than if it were moving constantly at that velocity. This is due to the fact that the fluid boundary layer will be thinner than in the steady-state case, due to the delay associated with acceleration of the surrounding fluid. The additional drag due to acceleration of the particle relative to the fluid is the virtual-mass term.

In a suspension the same effect will occur, but there is no information available at present on the correct form for the virtual-mass expression. It is usually assumed (*6–8*) to be given by the product of some function of the solids concentration (the virtual-mass coefficient) and the rate of change of the solid-liquid relative velocity. However, there is no knowledge of the way in which the virtual-mass coefficient varies with concentration, nor is it clear which is the correct form for the rate of change of the relative velocity. Murray (*6*) mentions two forms, both of which are symmetrical in v and w; that is, interchanging v and w only changes the sign. It seems that the virtual-mass term must have this property since it must act equally but oppositely on the solid and liquid phases. Using the form which Murray chose for his analysis of fluidization, a virtual-mass term in the form

$$-C\left\{\left(\frac{\partial(v-w)}{\partial t}\right)_x + (v-w)\left(\frac{\partial(v-w)}{\partial x}\right)_t\right\}$$

is added to Eq. (7), where virtual-mass coefficient C is a positive function of f.

Thus Eq. (7) becomes

$$\rho\frac{dv}{dt} = F_g - F_d - \frac{1}{f}\left(\frac{\partial\tau}{\partial x}\right)_t + \rho\left(\frac{dx}{dt} - v\right)\left(\frac{\partial v}{\partial x}\right)_t - C\left\{\left(\frac{\partial u^*}{\partial t}\right)_x + u^*\left(\frac{\partial u^*}{\partial x}\right)_t\right\} \tag{14}$$

and the discontinuity equation (Eq. 11) becomes

$$\tau_2 - \tau_1 = \rho\left\{\frac{f_1f_2}{f_2 - f_1}(v_2 - v_1)^2 - \frac{C_1f_1u_1^* + f_2C_2u_2^*}{2}(u_2^* - u_1^*)\right\} \tag{15}$$

This equation is more complex than Eq. (11), but it still demonstrates that a discontinuity cannot occur between two concentrations which are both in free settling. All discontinuities predicted by various analyses have an increase in concentration associated with a decrease in velocity (v, u, and u^*). Thus both terms in the braces in Eq. (15) have the same sign and so the right-hand side of Eq. (15) cannot be zero, which is required if both sides of the discontinuity are in free settling ($\tau_2 = \tau_1 = 0$).

The same conclusion is reached if the other symmetrical form suggested by Murray for the virtual-mass term is used; namely,

$$-C\left\{\left(\frac{\partial v}{\partial t}\right)_s - \left(\frac{\partial w}{\partial t}\right)_l\right\}$$

where

$$\left(\frac{\partial}{\partial t}\right)_l = \left(\frac{\partial}{\partial t}\right)_x + w\left(\frac{\partial}{\partial x}\right)_t$$

SYMBOLS

Positive direction is downward for all vector quantities

Types of plane: Eulerian—fixed relative to the container
Lagrangian—moving with the local solids velocity
Kynchian—fixed concentration

- f solids concentration, volume fraction, dimensionless
- F_g $g(\rho - \rho_l)$ = net gravitational force per unit volume acting on the solids, N/m^3
- F_d liquid-drag force per unit volume acting on the solids, N/m^3
- g acceleration due to gravity, m/sec^2
- r solids dilution, dimensionless $= 1/f$
- s total volume of solids per unit cross-sectional area, below a reference Lagrangian plane, m
- t time, sec
- u velocity of solids relative to slurry volume-average velocity, m/sec $= v - \phi_t$
- u^* velocity of solids relative to liquid, m/sec $= v - w$
- v velocity of solids relative to container, m/sec
- w velocity of liquid relative to container, m/sec
- x distance below reference Eulerian plane, m

Greek

ρ solids density, kg/m^3
ρ_l liquid density, kg/m^3
τ solids compressive stress based on total cross section, N/m^2
ϕ volumetric flux of solids, m/sec
ϕ_l volumetric flux of liquid, m/sec
ϕ_r solids volumetric flux relative to flux induced by the bulk flow, m/sec $= \phi - f\phi_t$
ϕ_t total volumetric flux, m/sec = volume-average velocity

Subscripts

1 above a discontinuity
2 below a discontinuity

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